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Characterising Pairs of Refinable Splines

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A complete characterisation is given, in terms of Fourier transforms, of pairs of refinable univariate spline functions, with knots at the integers, whose integer translates form a Riesz basis. © 1997 Academic Press

1. INTRODUCTION

One of the elegant and useful properties of spline functions with uniform knots is that they are refinable, i.e., they are linear combinations of translates of dilates of themselves. This property results in efficient subdivision algorithms for their evaluation and allows the construction of spline wavelets. In their recent paper [6], Lawton *et al.* characterise which spline functions with compact support are refinable and show, in particular, that the integer translates of such a function can form a Riesz basis if and only if the function is a uniform B-spline.

In [6] it is assumed that we have only one refinable function. However, the construction of wavelets from refinable splines has been extended to more than one refinable function [3]. In this paper we consider pairs of refinable splines with integer knots. This is radically different from the case of only one refinable function since there are pairs of refinable spline functions whose integer translates from Riesz bases but which have different continuity conditions than the usually considered B-splines. Indeed, to our knowledge, these basis functions have not been considered before. If we allow non-integer knots, then there is still greater flexibility and in [1, 8] this freedom has been used to construct refinable splines whose integer translates are orthonormal. This possibility is not considered here.

The natural tool for analysing refinable functions is the Fourier transform. In Section 2 we classify the Fourier transforms of any compactly supported splines and then show that refinability forces the Fourier transforms to take a particular form involving two trigonometric polynomials.

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We than give in Section 3 explicit characterisations of such spline functions whose integer translates form a Riesz basis, and verify that they span the space of all spline functions with certain continuity conditions at the integers. Finally, in Section 4, we show how our characterisation gives, as a special case, the usual B-splines with double knots at the integers.

2. REFINABLE SPLINES

We shall first characterise the Fourier transform of a spline function. The Fourier transform of a function ϕ will be denoted by $\hat{\phi}$ and is defined by

$$\hat{\phi}(u) := \int_{-\infty}^{\infty} \phi(x)\, e^{2\pi i x u}\, dx.$$

LEMMA 1. *Let ϕ be a spline function of degree n with knots $t_0 < \cdots < t_m$ and support $[t_0, t_m]$ and for $j = 0, ..., m$, $k = 0, ..., n$, let $a_{j,k} = \phi^{(k)}(t_j^+) - \phi^{(k)}(t_j^-)$. Then $\hat{\phi}$ is continuous, and for $u \neq 0$, $\hat{\phi}(u) = \psi(u)$, where*

$$\psi(u) = \sum_{j=0}^{m} \sum_{k=0}^{n} a_{j,k} \frac{e^{2\pi i u t_j}}{(-2\pi i u)^{k+1}}. \tag{2.1}$$

Conversely, if ψ is continuous and given by (2.1) for $u \neq 0$ and constants $a_{j,k}$ and $t_0 < \cdots < t_m$, then $\psi = \hat{\phi}$, where ϕ is a spline function as above.

Proof. Since ϕ is L^1, $\hat{\phi}$ is continuous. That $\hat{\phi} = \psi$ as given by (2.1) follows by integration by parts, as pointed out in [6].

Conversely, suppose that ψ is continuous and given by (2.1) for $u \neq 0$. Let ϕ be the spline function with support on $[t_0, t_m]$ satisfying

$$\phi^{(k)}(t_j^+) - \phi^{(k)}(t_j^-) = a_{j,k}, \qquad j = 0, ..., m-1, k = 0, ..., n.$$

Then by the first part of Lemma 1,

$$\hat{\phi}(u) = \psi(u) + \sum_{k=0}^{n} b_k \frac{e^{2\pi i u t_m}}{(-2\pi i u)^{k+1}}, \qquad u \neq 0,$$

for constants $b_0, ..., b_n$. Since both $\hat{\phi}$ and ψ are continuous at $u = 0$, we must have $b_0 = \cdots = b_n = 0$. ∎

Now let ϕ_1, ϕ_2 be compactly supported spline functions of degree n with knots at the integers. We suppose that $\phi = (\phi_1, \phi_2)^T$ is m-refinable; i.e., for some integer $m \geqslant 2$, ϕ satisfies a refinement equation

$$\phi(x) = \sum_{j=-N}^{N} A(j)\, \phi(mx - j), \qquad x \in R, \tag{2.2}$$

where for $j=-N, ..., N$, $A(j)$ is a 2×2 matrix. By Lemma 1 we see that

$$\hat{\phi}(u)=\sum_{j=0}^{n} u^{-j-1}T_j(u), \qquad u\neq 0, \tag{2.3}$$

where for $j=0, ..., n$,

$$T_j(u)=\sum a_{l,j}e^{2\pi ilu}, \tag{2.4}$$

where $a_{l,j}\in R^2$ and the summation is over some finite sets of integers 1.

Taking Fourier transforms of (2.2) gives

$$\hat{\phi}(u)=\frac{1}{m}\,C(e^{2\pi iu/m})\,\hat{\phi}\left(\frac{u}{m}\right), \tag{2.5}$$

where

$$C(z)=\sum_{j=-N}^{N} A(j)\,z^j, \qquad |z|=1. \tag{2.6}$$

Substituting (2.3) into (2.5) and equating powers of u gives, as in [6],

$$T_j(u)=m^jC(e^{2\pi iu/m})\,T_j\left(\frac{u}{m}\right), \qquad j=0, ..., n. \tag{2.7}$$

We now show that there are only two non-zero terms in the summation in (2.3).

LEMMA 2. *If ϕ_1, ϕ_2 are compactly supported spine functions of degree n with knots at the integers and $\phi=(\phi_1, \phi_2)^{\mathrm{T}}$ is m-refinable, then for some $0\leqslant j\leqslant k\leqslant n$,*

$$\hat{\phi}(u)=u^{-j-1}T_j(u)+u^{-k-1}T_k(u), \qquad u\neq 0, \tag{2.8}$$

where T_j and T_k are trigonometric polynomials as in (2.4).

Proof. If only one of the trigonometric polynomials $T_0, ..., T_n$ in (2.3) does not vanish identically, then (2.8) certainly holds. So we may suppose that for some $0\leqslant j\leqslant k\leqslant n$, T_j and T_k do not vanish identically.

We claim that there is some u for which the 2-vectors $T_j(u)$ and $T_k(u)$ are linearly independent. Suppose this is not the case. Then there is a scalar function $\lambda(u)$ such that whenever $T_j(u)\neq 0$,

$$T_k(u)=\lambda(u)\,T_j(u). \tag{2.9}$$

Now let $T_j(u)=(e^{2\pi iu}-1)^l\,\tilde{T}_j(u)$, $T_k(u)=(e^{2\pi iu}-1)^P\,\tilde{T}_k(u)$, where $\tilde{T}_j(0)\neq 0\neq\tilde{T}_k(0)$ and let I be a neighbourhood of 0 on which $\tilde{T}_j$ and $\tilde{T}_k$ do not vanish. From (2.7),

$$T_j(u)=m^jC(e^{2\pi iu/m})\;T_j\left(\frac{u}{m}\right),$$

$$T_k(u)=m^kC(e^{2\pi iu/m})\;T_k\left(\frac{u}{m}\right),$$

and applying (2.9) gives

$$m^kC(e^{2\pi iu/m})\;\lambda\left(\frac{u}{m}\right)\;T_j\left(\frac{u}{m}\right)=\lambda(u)\;m^jC(e^{2\pi iu/m})\;T_j\left(\frac{u}{m}\right).$$

For u in I with $u\neq 0$ we have $T_j(u)\neq 0$ and so $C(e^{2\pi iu/m})\;T_j(u/m)\neq 0$. Thus

$$\begin{aligned} m^k\lambda\left(\frac{u}{m}\right)&=m^j\lambda(u),\\ \lambda\left(\frac{u}{m}\right)&=m^{j-k}\lambda(u). \end{aligned} \tag{2.10}$$

Hence $\lim_{u\to 0}\lambda(u)=0$ and so $p>l$. Thus we can suppose

$$\lambda(u)=(e^{2\pi iu}-1)^{p-l}\,\frac{\tilde{T}_k(u)}{\tilde{T}_j(u)},\qquad u\in I. \tag{2.11}$$

Differentiating (2.10) gives

$$\lambda^{(r)}\left(\frac{u}{m}\right)=m^{j-k+r}\lambda^{(r)}(u),\qquad r=0,1,...,u\in I. \tag{2.12}$$

Putting $r=k-j+1$ and applying (2.12) s times gives

$$\lambda^{(k-j+1)}\left(\frac{u}{m^s}\right)=m^s\lambda^{(k-j+1)}(u),\qquad u\in I. \tag{2.13}$$

If $\lambda^{(k-j+1)}(u)\neq 0$ for any u in I, then (2.13) would contradict λ being analytic at $u=0$. Thus $\lambda^{(k-j+1)}$ vanishes identically on I so λ is an algebraic polynomial of degree $k-j$, which contradicts (2.11).

So there is some u such that $T_j(u)$ and $T_k(u)$ are linearly independent. Let T denote the 2×2 matrix $[T_jT_k]$. Then $\det T$ is a trigonometric polynomial, and since it does not vanish identially, there is some neighbourhood J of 0 such that T is non-singular for u in J, $u \neq 0$. Now by (2.7),

$$T(u) = C(e^{2\pi iu/m})\, T\left(\frac{u}{m}\right)\begin{bmatrix} m^j & 0 \\ 0 & m^k \end{bmatrix}$$
$$= C(e^{2\pi iu/m})\ldots C(e^{2\pi iu/m^r})\, T\left(\frac{u}{m^r}\right)\begin{bmatrix} m^{rj} & 0 \\ 0 & m^{rk} \end{bmatrix},$$

for $r = 1, 2, \ldots$. Also from (2.5), for $r = 1, 2, \ldots$,

$$\hat{\phi}(u) = \frac{1}{m^r}\, C(e^{2\pi iu/m})\ldots C(e^{2\pi iu/m^r})\, \hat{\phi}\left(\frac{u}{m^r}\right)$$

and so for u in J, $u \neq 0$,

$$\hat{\phi}(u) = \frac{1}{m^r}\, T(u)\begin{bmatrix} m^{-rj} & 0 \\ 0 & m^{-rk} \end{bmatrix} T\left(\frac{u}{m^r}\right)^{-1} \hat{\phi}\left(\frac{u}{m^r}\right).$$

Hence

$$\lim_{r\to\infty}\begin{bmatrix} m^{-r(j+1)} & 0 \\ 0 & m^{-r(k+1)} \end{bmatrix} T\left(\frac{u}{m^r}\right)^{-1} \hat{\phi}\left(\frac{u}{m^r}\right) = T(u)^{-1}\,\hat{\phi}(u). \tag{2.14}$$

Expanding in a Taylor series about $u = 0$ we can write

$$T(u)^{-1}\,\hat{\phi}(u) = (e^{2\pi iu} - 1)^{-q}\begin{bmatrix} au^{\alpha} + 0(u^{\alpha+1}) \\ bu^{\beta} + 0(u^{\beta+1}) \end{bmatrix}$$

for some non-negative integers q, α, β, and non-zero constants a, b. Then from (2.14),

$$T(u)^{-1}\,\hat{\phi}(u) = \lim_{r\to\infty}(e^{2\pi iu/m^r} - 1)^{-q}\begin{bmatrix} au^{\alpha} m^{-r(j+1+\alpha)} \\ bu^{\beta} m^{-r(k+1+\beta)} \end{bmatrix}$$
$$= (2\pi i)^{-q}\lim_{r\to\infty}\begin{bmatrix} au^{\alpha-q} m^{-r(j+1+\alpha-q)} \\ bu^{\beta-q} m^{-r(k+1+\beta-q)} \end{bmatrix}.$$

We must have either $\lim_{r\to\infty} u^{\alpha-q} m^{-r(j+1+\alpha-q)} = 0$ or $j+1+\alpha-q=0$, in which case $\lim_{r\to\infty} u^{\alpha-q} m^{-r(j+1+\alpha-q)} = u^{-j-1}$. There is a similar result for the second row and hence

$$\hat{\phi}(u) = T(u)\begin{bmatrix} cu^{-j-1} \\ du^{-k-1} \end{bmatrix} = cu^{-j-1}T_j(u) + du^{-k-1}T_k(u)$$

for some constants c and d. Comparison with (2.3) gives the result. ∎

We note that by Lemma 1 the functions ϕ_1, ϕ_2 given by (2.8) have discontinuities only in the derivatives of order j and k. Since ϕ_1 and ϕ_2 have compact support, they have exact degree j or k depending on whether or not the corresponding component of T_k vanishes identically.

3. RIESZ BASES

In addition to the conditions on ϕ in the last section we now impose the condition that the integer translates of ϕ_1, ϕ_2 form a Riesz basis, i.e., there are constants $A>0$, $B>0$ such that for any $a=(a_j)_{-\infty}^{\infty}$ in l^2,

$$A\|a\|_{l^2} \leqslant \left\| \sum_{-\infty}^{\infty} a_{2j}\phi_1(\cdot - j) + \sum_{-\infty}^{\infty} a_{2j+1}\phi_2(\cdot - j) \right\|_{L^2} \leqslant B\,\|a\|_{l^2}.$$

We shall use the following result, which is a special case of Theorem 5.1 of [5] and Lemma 3 of [2].

LEMMA A. *If ϕ_1, ϕ_2 in $L^1(\mathbb{R})$ have compact support, then the integer translates of ϕ_1, ϕ_2 form a Riesz basis if and only if the vectors $(\hat{\phi}_1(u+r))_{r=-\infty}^{\infty}$ and $(\hat{\phi}_2(u+r))_{r=-\infty}^{\infty}$ are linearly independent for all u.*

We now state our main result. Throughout the rest of the paper we shall write $z=e^{2\pi i u}$.

THEOREM 1. *For an integer $k \geqslant 2$ and functions ϕ_1, ϕ_2 the following are equivalent:*

(a) *The functions ϕ_1, ϕ_2 are compactly supported spline functions of degree $k-1$ with knots at the integers and at least one of ϕ_1, ϕ_2 has exact degree $k-1$. Moreover, the integer translates of ϕ_1, ϕ_2 form a Riesz basis and $\phi=(\phi_1, \phi_2)^T$ is m-refinable for some $m \geqslant 2$.*

(b) *For an integer j, $1 \leqslant j \leqslant k-1$,*

$$\begin{bmatrix} \hat{\phi}_1(u) \\ \hat{\phi}_2(u) \end{bmatrix} = M(z) \begin{bmatrix} 1 & -p(z) \\ 0 & (z-1)^k \end{bmatrix} \begin{bmatrix} (2\pi i u)^{-j} \\ (2\pi i u)^{-k} \end{bmatrix}, \tag{3.1}$$

where $p(z)$ is the Taylor polynomial of degree $k-1$ at $z=1$ for $(\log z)^{k-j}$ and M is a 2×2 matrix of Laurent polynomials with $\det M(z) = cz^l$ *for some constant $c \neq 0$ and integer l.*

(c) *For an integer j, $1 \leqslant j \leqslant k-1$,*

$$\begin{bmatrix} \hat{\phi}_1(u) \\ \hat{\phi}_2(u) \end{bmatrix} = L(z) \begin{bmatrix} -q(z) & (z-1)^{k-j} \\ (z-1)^j & 0 \end{bmatrix} \begin{bmatrix} (2\pi i u)^{-j} \\ (2\pi i u)^{-k} \end{bmatrix}, \tag{3.2}$$

where L is of the same form as M in (3.1) *and* $q(z)$ *is the Taylor polynomial of degree* $j-1$ *at* $z=1$ *for* $(z-1/\log z)^{k-j}$.

Proof. Suppose that (a) is satisfied. Then by Lemma 2, we may write

$$\hat{\phi}(u)=(2\pi iu)^{-j}\,P(z)+(2\pi iu)^{-k}\,Q(z) \tag{3.3}$$

for some j, $1\leqslant j\leqslant k-1$, and Laurent polynomials P and Q. Since $\hat{\phi}(0)$ is finite, $(2\pi iu)^{k-j}\,P(z)+Q(z)$ has a zero of order k at $u=0$, i.e., $(\log z)^{k-j}P(z)+Q(z)$ has a zero of order k at $z=1$. Defining p as in (b) we see that $(\log z)^{k-j}-p(z)$ also has a zero of order k at $z=1$ and thus so does

$$(\log z)^{k-j}\,P(z)+Q(z)-((\log z)^{k-j}-p(z))\,P(z)=Q(z)+p(z)\,P(z). \tag{3.4}$$

Hence we can write

$$p(z)\,P(z)+Q(z)=(z-1)^k\,R(z) \tag{3.5}$$

for some Laurent polynomial $R(z)$. So from (3.3),

$$\begin{aligned}\hat{\phi}(u)&=[P(z)\;Q(z)]\begin{bmatrix}(2\pi iu)^{-j}\\(2\pi iu)^{-k}\end{bmatrix}\\&=[P(z)-p(z)\,P(z)+(z-1)^k\,R(z)]\begin{bmatrix}(2\pi iu)^{-j}\\(2\pi iu)^{-k}\end{bmatrix}\\&=[P(z)\;R(z)]\begin{bmatrix}1&-p(z)\\0&(z-1)^k\end{bmatrix}\begin{bmatrix}(2\pi iu)^{-j}\\(2\pi iu)^{-k}\end{bmatrix}.\end{aligned} \tag{3.6}$$

To establish (b) it remains only to show that $M(z):=[P(z)\,R(z)]$ has determinant of form cz^{ℓ}. From (3.3) and Lemma A we know that the two components of

$$P(z)(2\pi i)^{-j}\,((u+r)^{-j})_{r=-\infty}^{\infty}+Q(z)(2\pi i)^{-k}((u+r)^{-k})_{r=-\infty}^{\infty} \tag{3.7}$$

are linearly independent for all u. Thus for $0<u<1$, the matrix $[P(z)\;Q(z)]$ is non-singular and so $[P(z)\;R(z)]$ is non-singular.

We next consider the case $u=0$. Putting $u=0$, $r=0$, in (3.7) gives

$$\begin{aligned}&\lim_{u\to 0}\,\{P(z)(2\pi iu)^{-j}+Q(z)(2\pi iu)^{-k}\}\\&\quad=\lim_{z\to 1}\left\{\frac{(\log z)^{k-j}-p(z)}{(\log z)^k}\,P(z)+\frac{(z-1)^k}{(\log z)^k}\,R(z)\right\}\\&\quad=aP(1)+R(1)\end{aligned}$$

for some constant $a \neq 0$. Since $Q(1)=0$, (3.7) becomes

$$\begin{cases} P(1)(2\pi i r)^{-j} & r \neq 0 \\ aP(1)+R(1), & r=0. \end{cases} \tag{3.8}$$

Since the two components of (3.8) are linearly independent, the matrix $[P(1)\ R(1)]$ is non-singular. Now from (3.3) and (2.5) we have

$$[P(e^{2\pi iu})\ Q(e^{2\pi iu})] = C(e^{2\pi iu/m})[m^{j-1}P(e^{2\pi iu/m})\ m^{k-1}Q(e^{2\pi iu/m})]$$

and so $\det[P(z)\ Q(z)]$ is a factor of $\det[P(z^m)\ Q(z^m)]$. (In the terminology of [3], $\det[P(z)\ Q(z)]$ is m-closed.) It follows that $\det[P(z)\ Q(z)]$ has its zeros on the unit circle or at $z=0$. But

$$\det[P(z)\ Q(z)] = (z-1)^k \det[P(z)\ R(z)]$$

and we know that $\det[P(z)\ R(z)]$ is non-zero on the unit circle. Thus $\det[P(z)\ R(z)]$ can vanish only at $z=0$, i.e., it has the form cz^l, which establishes (b).

We now assume that (b) holds and prove (a). Since $(2\pi iu)^{-j} - p(z)(2\pi iu)^{-k} = ((\log z)^{k-j} - p(z))(2\pi iu)^{-k}$ which is well-defined at $u=0$, $\hat{\phi}_1(0)$ and $\hat{\phi}_2(0)$ are well-defined and, by Lemma 1, ϕ_1 and ϕ_2 are compactly supported spline functions of degree $k-1$ with knots at the integers. Moreover, since $M(z)$ is non-singular on the unit circle, at least one of ϕ_1 and ϕ_2 has exact degree $k-1$.

We write $M(z) = [P(z)\ R(z)]$ and $M(z)\left[\begin{smallmatrix} 1 & -p(z) \\ 0 & (z-1)^k \end{smallmatrix}\right] = [P(z)\ Q(z)]$. Since $\det M(z) = cz^l$, $c \neq 0$, $M(z)$ is non-singular whenever $|z|=1$. So the two components of (3.7) are linearly independent for all u in (0,1). Moreover, the two components of (3.8) are linearly independent and so the two components of (3.7) are also linearly independent for $u=0$. Thus by Lemma A, the integer translates of ϕ_1 and ϕ_2 form a Riesz basis.

It remains to show that $\phi = (\phi_1, \phi_2)^{\mathrm{T}}$ is m-refinable. Now

$$\begin{bmatrix} 1 & -p(z^m) \\ 0 & (z^m-1)^k \end{bmatrix} = C(z) \begin{bmatrix} 1 & -p(z) \\ 0 & (z-1)^k \end{bmatrix} \begin{bmatrix} m^{j-1} & 0 \\ 0 & m^{k-1} \end{bmatrix} \tag{3.9}$$

where

$$C(z) := \begin{bmatrix} m^{1-j} & \dfrac{m^{k-j}(p(z)-p(z^m))}{m^{k-1}(z-1)^k} \\ 0 & \dfrac{(z^m-1)^k}{m^{k-1}(z-1)^k} \end{bmatrix}.$$

We note that, for $v=0, ..., k-1$,

$$\begin{aligned}\frac{\partial^v}{\partial z^v}\, p(z^m)\Big|_{z=1} &= \frac{\partial^v}{\partial z^v}\log(z^m)^{k-j}\Big|_{z=1}\\ &= m^{k-j}\,\frac{\partial^v}{\partial z^v}(\log z)^{k-j}\Big|_{z=1}\\ &= m^{k-j}p^{(v)}(1).\end{aligned}$$

Thus $m^{k-j}p(z)-p(z^m)$ is divisible by $(z-1)^k$ and so $C(z)$ is a matrix of polynomials. By (3.9),

$$\begin{aligned}&M(z^m)\begin{bmatrix}1 & -p(z^m)\\ 0 & (z^m-1)^k\end{bmatrix}\\ &\quad = M(z^m)\, C(z)\, M(z)^{-1}\, M(z)\begin{bmatrix}1 & -p(z)\\ 0 & (z-1)^k\end{bmatrix}\begin{bmatrix}m^{j-1} & 0\\ 0 & m^{k-1}\end{bmatrix},\end{aligned}$$

and so by (3.1),

$$\begin{bmatrix}\hat{\phi}_1(mu)\\ \hat{\phi}_2(mu)\end{bmatrix} = \frac{1}{m}\, M(z^m)\, C(z)\, M(z)^{-1}\begin{bmatrix}\hat{\phi}_1(u)\\ \hat{\phi}_2(u)\end{bmatrix}. \tag{3.10}$$

Since $\det M(z)=cz^{\ell}$, $c\neq 0$, $M(z)^{-1}$ is a matrix of Laurent polynomials and so (3.10) shows that ϕ is m-refinable, which establishes (a).

Next we shall show that (b) implies (c). Assume that (b) holds and write $M(z)=[P(z)\ R(z)]$, $Q(z)=-p(z)\,P(z)+(z-1)^k R(z)$. Then

$$(\log z)^{k-j}\,P(z)+Q(z)=((\log z)^{k-j}-p(z))\,P(z)+(z-1)^k\,R(z),$$

which has a zero of order k at $z=1$. Thus $Q(z)$ has a zero of order $k-j$ at $z=1$ and we may write

$$Q(z)=(z-1)^{k-j}\,U(z) \tag{3.11}$$

for some Laurent polynomial $U(z)$. Then

$$\begin{aligned}P(z)+q(z)\,U(z) &= (\log z)^{j-k}\,((\log z)^{k-j}\,P(z)+Q(z))\\ &\quad+\left(q(z)-\left(\frac{z-1}{\log z}\right)^{k-j}\right)U(z)\end{aligned} \tag{3.12}$$

which has a zero of order j at $z=1$. So for some Laurent polynomial $V(z)$,

$$P(z)+q(z)\,U(z)=(z-1)^j\,V(z). \tag{3.13}$$

Hence

$$[P(z)\ R(z)]\begin{bmatrix}1 & -p(z)\\ 0 & (z-1)^k\end{bmatrix}=[P(z)\ Q(z)]$$

$$=[U(z)\ V(z)]\begin{bmatrix}-q(z) & (z-1)^{k-j}\\ (z-1)^j & 0\end{bmatrix}, \tag{3.14}$$

by (3.11) and (3.13). Thus (3.1) gives (3.2) with $L(z)=[U(z)\ V(z)]$ and $\det L(z)=-\det M(z)$.

Finally, we show that (c) implies (b). Assume that (c) holds and write $L(z)=[U(z)\ V(z)]$. Define $[P(z)\ Q(z)]$ by (3.14). Then (3.11) and (3.13) hold. From (3.13), $P(z)+q(z)\,U(z)$ has a zero of order j at $z=1$ and so by (3.12), $(\log z)^{k-j}\,P(z)+Q(z)$ has a zero of order k at $z=1$ and we may write (3.5) for a Laurent polynomial $R(z)$. Thus

$$[P(z)\ Q(z)]=[P(z)\ R(z)]\begin{bmatrix}1 & -p(z)\\ 0 & (z-1)^k\end{bmatrix}$$

and (3.1) holds with $M(z)=[P(z)\ R(z)]$ and $\det M(z)=-\det L(z)$. ∎

Remarks. 1. The proof of Theorem 1 shows that a refinement equation of form (2.2) has solutions satisfying (a) of Theorem 1 if and only if the "symbol" $C(z)$ in (2.5) satisfies the following. For an integer j, $1\leqslant j\leqslant k-1$,

$$C(z)=M(z^m)\begin{bmatrix} m^{1-j} & \dfrac{m^{k-j}p(z)-p(z^m)}{m^{k-1}(z-1)^k}\\ 0 & \dfrac{(z^m-1)^k}{m^{k-1}(z-1)^k}\end{bmatrix}M(z)^{-1},$$

where p and M are as in (b) of Theorem 1.

2. By taking suitable integer translates of ϕ_1 and/or ϕ_2 we may assume that $\det M$ and $\det L$ are constants.

3. The functions ϕ_1 and ϕ_2 have discontinuities only in the derivatives or order $j-1$ and $k-1$. In the choice (3.1) with M the identity, ϕ_2 is the usual B-spline of degree $k-1$ with simple knots and support $[0,k]$. Here ϕ_1 has support on $[0,k-1]$ with simple knots at $1,\ldots,k-1$, while at 0 it has discontinuities in derivatives of order $j-1$ and of order $k-1$. In the choice (3.2) with L the identity, ϕ_2 is the usual B-spline of degree $j-1$ with simple knots and support $[0,j]$. Here ϕ_1 has support on $[0,\max\{j-1,k-j\}]$. If $2j\leqslant k$, then for $j\leqslant \ell\leqslant k-j$, ϕ_1 has a discontinuity only in the derivative of order $k-1$ at ℓ. If $2j\geqslant k+2$, then ϕ_1 has degree $j-1$ on $[k-j,j-1]$.

4. The choice of M as the identity in (3.1) gives the minimum number of discontinuities in the derivative of order $j-1$, while the choice of L as the identity in (3.2) gives the minimum number of discontinuities in the derivatives of order $k-1$. By choosing suitable matrices M in (2.1) (or L in (2.2)) we can get intermediate cases, as illustrated in the following examples.

EXAMPLE 1: $j=1$. Here (3.1) with M as the identity becomes

$$\begin{bmatrix} 1 & -(z-1)^{k-1} \\ 0 & (z-1)^k \end{bmatrix}, \tag{3.15}$$

where we shall omit the term $[(2\pi iu)^{-j}(2\pi iu)^{-k}]^{\mathrm{T}}$. Then

$$\begin{bmatrix} -1 & 0 \\ z & 1 \end{bmatrix} \begin{bmatrix} 1 & -(z-1)^{k-1} \\ 0 & (z-1)^k \end{bmatrix} = \begin{bmatrix} -1 & (z-1)^{k-1} \\ z & -(z-1)^{k-1} \end{bmatrix} \tag{3.16}$$

and

$$\begin{bmatrix} -1 & 0 \\ z-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -(z-1)^{k-1} \\ 0 & (z-1)^k \end{bmatrix} = \begin{bmatrix} -1 & (z-1)^{k-1} \\ z-1 & 0 \end{bmatrix},$$

which is the case (3.2) with L as the identity.

EXAMPLE 2: $k=4$, $j=2$. Here (3.1) with M as the identity gives

$$A := \begin{bmatrix} 1 & (z-2)(z-1)^2 \\ 0 & (z-1)^4 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 & 0 \\ -z & 1 \end{bmatrix} A = \begin{bmatrix} 1 & (z-2)(z-1)^2 \\ -z & (z-1)^2 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -z \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -z & 1 \end{bmatrix} A = \begin{bmatrix} 1+z^2 & -2(z-1)^2 \\ -2z & 2(z-1)^2 \end{bmatrix}, \tag{3.17}$$

and

$$\begin{bmatrix} 0 & 1 \\ 1 & 2-z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -z & 1 \end{bmatrix} A = \begin{bmatrix} -z & (z-1)^2 \\ (z-1)^2 & 0 \end{bmatrix},$$

which is the case (3.2) with L as the identity.

For applications such as computer-aided design it is sometimes useful to choose a basis which forms a partition of unity, i.e.,

$$\sum_{l=-\infty}^{\infty} (\phi_1+\phi_2)(x-l)=1, \qquad x\in R. \tag{3.18}$$

In fact, the integer translates of the functions ϕ_1, ϕ_2 given by either (3.16) or (3.17) form a partition of unity, which follows from our next result.

THEOREM 2. *The integer translates of the functions* ϕ_1, ϕ_2 *given by* (3.1) *form a partition of unity if and only if*

$$[1\ \ 1]\, M(1)=[0\ \ 1].$$

Proof. Equation (3.18) is satisfied if and only if

$$\hat{\phi}_1(l)+\hat{\phi}_2(l)=\delta_{l,0}, \qquad l\in Z. \tag{3.19}$$

Let $[1\ \ 1]\, M(z)=[A(z)\ B(z)]$. Then from (3.1),

$$\hat{\phi}_1(u)+\hat{\phi}_2(u)=A(z)\{(2\pi iu)^{-j}-p(z)(2\pi iu)^{-k}\}+B(z)(z-1)^k\,(2\pi iu)^{-k}. \tag{3.20}$$

For l in Z, $l\neq 0$, recalling $p(1)=0$ gives

$$\hat{\phi}_1(l)+\hat{\phi}_2(l)=A(1)(2\pi il)^{-j}. \tag{3.21}$$

So if (3.19) is satisfied, then $A(1)=0$ and (3.20) gives

$$\hat{\phi}_1(0)+\hat{\phi}_2(0)=B(1),$$

which gives $B(1)=1$. Conversely, if $A(1)=0, B(1)=1$, then (3.20) gives (3.19). ∎

Now for $1\leqslant j\leqslant k-1$, we denote by $S_{j,k}$ the set of all spline functions of degree $k-1$ with knots at the integers and discontinuities only in derivates of order $j-1$ and $k-1$. We know that the functions ϕ_1, ϕ_2 of (b) or (c) in Theorem 1 are such that their integer translates form a Riesz basis. Our next result shows, in particular, that they form a Riesz basis for $S_{j,k}\cap L^2(R)$.

THEOREM 3. *Let* ϕ_1, ϕ_2 *be as in* (b) *or* (c) *of Theorem* 1. *Then any* f *in* $S_{j,k}$ *can be expressed uniquely in the form*

$$f=\sum_{\ell=-\infty}^{\infty} a_\ell\phi_1(\cdot-\ell)+\sum_{\ell=-\infty}^{\infty} b_\ell\phi_2(\cdot-\ell), \tag{3.22}$$

for some constants $a_\ell, b_\ell, \ell\in Z$.

Proof. By Theorem 1, (b) and (c) are equivalent and so it suffices to assume that ϕ_1, ϕ_2 satisfy (b). We shall first consider the case when M is the identity. In this case ϕ_1 has a discontinuity in the derivative of order $j-1$ only at 0 and by Lemma 1, $\phi_1^{(j-1)}(0^+)-\phi_1^{(j-1)}(0^-)=(-1)^j$.

Take f in $S_{j,k}$ and suppose that for ℓ in Z, $f^{(j-1)}(\ell^+)-f^{(j-1)}(l^-)=(-1)^j a_\ell$. Let $g=f-\sum_{-\infty}^{\infty} a_\ell \phi_1(\cdot-\ell)$. Then $g^{(j-1)}$ has no discontinuities and so g is a spline function of degree $k-1$ with simple knots. Since ϕ_2 is a B-spline of degree $k-1$ with simple knots, we may write

$$g=\sum_{\ell=-\infty}^{\infty} b_\ell \phi_2(\cdot-\ell)$$

and so (3.22) holds. Now suppose that in (3.22), f is identically zero. Then for all ℓ in Z, $a_\ell=(-1)^j(f^{(j-1)}(\ell^+)-f^{(j-1)}(\ell^-))=0$. So $\sum_{-\infty}^{\infty} b_\ell \phi_2(\cdot-\ell)$ is identically zero and, by the linear independence of the B-splines $\phi_2(\cdot-\ell)$, $b_\ell=0$ for all ℓ in Z. So our representation (3.22) is unique.

Now suppose that ϕ_1, ϕ_2 are given by (b), for a general matrix M, as in Theorem 1. Letting $\tilde{\phi}_1, \tilde{\phi}_2$ denote the corresponding functions when M is replaced by the identity, (3.1) shows that ϕ_1, ϕ_2 are finite linear combinations of integer translates of $\tilde{\phi}_1$ and $\tilde{\phi}_2$. Since $M(z)^{-1}$ is also a matrix of Laurent polynomials, we also see that $\tilde{\phi}_1, \tilde{\phi}_2$ are finite linear combinations of integer translates of ϕ_1 and ϕ_2. We have shown above that any f in $S_{j,k}$ can be expressed in the form (3.22). We have also shown that the integer translates of $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are linearly independent and it follows that the integer translates of ϕ_1 and ϕ_2 are linearly independent; see Theorem 5.1 of [4]. ∎

Theorem 3 and Theorem 1 immediately give the following.

COROLLARY 1. *Suppose that ϕ_1, ϕ_2 are compactly supported spline functions with knots at the integers which are m-refinable for some $m \geqslant 2$. If their integer translates form a Riesz basis, then they are linearly independent.*

4. HERMITE SPLINES

In this section we consider the case $j=k-1$. Here a possible choice of ϕ_1, ϕ_2 is the usual B-splines with double knots at the integers. We denote the knots by t_ℓ, where

$$t_{2\ell}=t_{2\ell+1}=\ell, \qquad \ell \in Z, \tag{4.1}$$

and denote by N_ℓ^k the B-spline of degree $k-1$ with knots $t_\ell, ..., t_{\ell+k}$. Thus $N_{\ell+2}^k = N_\ell^k(\cdot - 1)$, $\ell \in Z$. The well-known recurrence relation for the derivatives gives, for $k \geqslant 2$,

$$N_\ell^{k+1'} = \frac{k}{t_{\ell+k} - t_\ell} N_\ell^k - \frac{k}{t_{\ell+k+1} - t_{\ell+1}} N_{\ell+1}^k,$$

and so

$$-2\pi i u \hat{N}_\ell^{k+1}(u) = \frac{k}{t_{\ell+k} - t_\ell} \hat{N}_\ell^k(u) - \frac{k}{t_{\ell+k+1} - t_{\ell+1}} \hat{N}_\ell^k(u). \tag{4.2}$$

From Lemma 1 we may write

$$\begin{bmatrix} \hat{N}_0^k(u) \\ \hat{N}_1^k(u) \end{bmatrix} = T^k(z) \begin{bmatrix} (2\pi i u)^{-k+1} \\ (2\pi i u)^{-k} \end{bmatrix}, \tag{4.3}$$

where T^k is a 2×2 matrix of polynomials. Substituting (4.3) into (4.2), and noting that $\hat{N}_2^k(u) = z\hat{N}_0^k(u)$, gives

$$T^{k+1}(z) = k \begin{bmatrix} -1 & 1 \\ z & -1 \end{bmatrix} \begin{bmatrix} (t_k - t_0)^{-1} & 0 \\ 0 & (t_{k+1} - t_1)^{-1} \end{bmatrix} T^k(z).$$

Recalling (4.1) we see that

$$T^{2r+1}(z) = 2 \begin{bmatrix} -1 & 1 \\ z & -1 \end{bmatrix} T^{2r}(z), \qquad r \geqslant 1, \tag{4.4}$$

$$T^{2r}(z) = (2r-1) \begin{bmatrix} -1 & 1 \\ z & -1 \end{bmatrix} \begin{bmatrix} (r-1)^{-1} & 0 \\ 0 & r^{-1} \end{bmatrix} T^{2r-1}(z), \qquad r \geqslant 2. \tag{4.5}$$

Direct calculation shows that

$$T^2(z) = \begin{bmatrix} -1 & z-1 \\ z & 1-z \end{bmatrix}. \tag{4.6}$$

The recurrence relation (4.4)–(4.6) gives $T^k(z)$ as a product of matrices, which gives $\hat{N}_0^k$, $\hat{N}_1^k$ by (4.3). For further information on these B-splines, see [7]. Now $\phi_1 = N_0^k$, $\phi_2 = N_1^k$ satisfy (a) of Theorem 1 and so we may write as in (3.1),

$$T^k(z) = M(z) \begin{bmatrix} 1 & -p(z) \\ 0 & (z-1)^k \end{bmatrix}. \tag{4.7}$$

For example, when $k=2$, we have $P(z)=z-1$ and from (4.6)

$$T^2(z)=\begin{bmatrix} -1 & 0 \\ z & 1 \end{bmatrix}\begin{bmatrix} 1 & 1-z \\ 0 & (z-1)^2 \end{bmatrix}. \tag{4.8}$$

(Note that Theorem 2 confirms that the integer translates of N_0^2 and N_1^2 form a partition of unity.) Our next result gives an explicit representation for M in (4.7).

THEOREM 4. *The B-splines* N_0^k, N_1^k *are given by*

$$\begin{bmatrix} \hat{N}_0^k(u) \\ \hat{N}_1^k(u) \end{bmatrix} = M^k(z)\begin{bmatrix} 1 & p_k(z) \\ 0 & (z-1)^k \end{bmatrix}\begin{bmatrix} (2\pi iu)^{-k+1} \\ (2\pi iu)^{-k} \end{bmatrix}, \tag{4.9}$$

where $p_k(z)=\sum_{\ell=1}^{k-1}((1-z)^\ell/\ell)$ *and* M^k *is defined as follows. For* $r\geqslant 1$,

$$M^{2r}(z)=\binom{2r-1}{r}\begin{bmatrix} -r & 0 \\ 0 & 1 \end{bmatrix} A(1)\prod_{\ell=1}^{r-1} B\left(\frac{2\ell}{r^2-\ell^2}\right) A(2\ell+1), \tag{4.10}$$

$$M^{2r+1}(z)=\binom{2r}{r}\begin{bmatrix} -r & 0 \\ 0 & 1 \end{bmatrix}\left\{\prod_{\ell=1}^{r} B\left(\frac{2\ell-1}{(r+1-\ell)(r+\ell)}\right) A(2\ell)\right\}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.11}$$

where

$$A(\lambda)=\begin{bmatrix} 1 & 0 \\ \lambda z & 1 \end{bmatrix}, \qquad B(\lambda)=\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}.$$

Proof. The proof is by induction on k. By (4.3) and (4.8) we see that the result holds for $k=2$. Assume that it holds for $k=2r$, $r\geqslant 1$. By (4.3), (4.4), and (4.9),

$$\begin{bmatrix} \hat{N}_0^{2r+1}(u) \\ \hat{N}_1^{2r+1}(u) \end{bmatrix} = 2\begin{bmatrix} -1 & 1 \\ z & -1 \end{bmatrix} M^{2r}(z)\begin{bmatrix} 1 & p_{2r}(z) \\ 0 & (z-1)^{2r} \end{bmatrix}\begin{bmatrix} (2\pi iu)^{-2r} \\ (2\pi iu)^{-2r-1} \end{bmatrix}. \tag{4.12}$$

Now by straightforward calculation,

$$\begin{bmatrix} -1 & 1 \\ z & -1 \end{bmatrix}\begin{bmatrix} -r & 0 \\ 0 & 1 \end{bmatrix} A(1)=\begin{bmatrix} r & 0 \\ 0 & -1 \end{bmatrix} B\left(\frac{1}{r(r+1)}\right)\begin{bmatrix} 1 & \frac{1}{r+1} \\ (r+1)z & 1 \end{bmatrix}. \tag{4.13}$$

Also for $\ell = 1, ..., r-1$,

$$\begin{bmatrix} 1 & \dfrac{1}{r+\ell} \\ (r+\ell)z & 1 \end{bmatrix} B\left(\frac{2\ell}{r^2-\ell^2}\right) A(2\ell+1)$$

$$= A(2\ell) \begin{bmatrix} 1 & \dfrac{1}{r-\ell} \\ (r-\ell)z & 1 \end{bmatrix} A(2\ell+1)$$

$$= A(2\ell)\, B\left(\frac{2\ell+1}{(r-\ell)(r+\ell+1)}\right) \begin{bmatrix} 1 & \dfrac{1}{r+\ell+1} \\ (r+\ell+1)z & 1 \end{bmatrix}. \tag{4.14}$$

Thus by (4.10), (4.13), and (4.14),

$$2\begin{bmatrix} -1 & 1 \\ z & -1 \end{bmatrix} M^{2r}(z)$$

$$= \binom{2r}{r} \begin{bmatrix} r & 0 \\ 0 & -1 \end{bmatrix} \left\{ \prod_{\ell=1}^{r-1} B\left(\frac{2\ell-1}{(r+1-\ell)(r+\ell)}\right) A(2\ell) \right\}$$

$$\times B\left(\frac{2r-1}{2r}\right) \begin{bmatrix} 1 & \dfrac{1}{2r} \\ 2rz & 1 \end{bmatrix}. \tag{4.15}$$

But

$$\begin{bmatrix} 1 & \dfrac{1}{2r} \\ 2rz & 1 \end{bmatrix} = A(2r) \begin{bmatrix} 1 & \dfrac{1}{2r} \\ 0 & 1-z \end{bmatrix} \tag{4.16}$$

and

$$\begin{bmatrix} 1 & \dfrac{1}{2r} \\ 0 & 1-z \end{bmatrix} \begin{bmatrix} 1 & p_{2r}(z) \\ 0 & (z-1)^{2r} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & p_{2r+1}(z) \\ 0 & (z-1)^{2r+1} \end{bmatrix}. \tag{4.17}$$

So combining (4.12), (4.15), (4.16), and (4.17) gives (4.9) for $k = 2r+1$ with M^k given by (4.11).

Now assume that the result holds for $k=2r-1$, $r\geqslant 2$. Then recalling (4.5),

$$\begin{bmatrix} \hat{N}_0^{2r}(u) \\ \hat{N}_1^{2r}(u) \end{bmatrix} = (2r-1)\begin{bmatrix} -1 & 1 \\ z & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{r-1} & 0 \\ 0 & \frac{1}{r} \end{bmatrix} M^{2r-1}$$

$$\times \begin{bmatrix} 1 & p_{2r-1}(z) \\ 0 & (z-1)^{2r-1} \end{bmatrix}\begin{bmatrix} (2\pi iu)^{-2r+1} \\ (2\pi iu)^{-2r} \end{bmatrix}. \tag{4.18}$$

Now

$$\begin{bmatrix} -1 & 1 \\ z & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{r-1} & 0 \\ 0 & \frac{1}{r} \end{bmatrix}\begin{bmatrix} -r+1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{r} \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{r} \\ rz & 1 \end{bmatrix}, \tag{4.19}$$

and for $\ell=1, ..., r-1$,

$$\begin{bmatrix} 1 & \frac{1}{r+\ell-1} \\ (r+\ell-1)z & 1 \end{bmatrix} B\left(\frac{2\ell-1}{(r-\ell)(r-1+\ell)}\right) A(2\ell)$$

$$= A(2\ell-1)\begin{bmatrix} 1 & \frac{1}{r-\ell} \\ (r-\ell)z & 1 \end{bmatrix} A(2\ell)$$

$$= A(2\ell-1)\, B\left(\frac{2\ell}{r^2-\ell^2}\right)\begin{bmatrix} 1 & \frac{1}{r+\ell} \\ (r+\ell)z & 1 \end{bmatrix}. \tag{4.20}$$

Thus by (4.11) (with r replaced by $r-1$), (4.19), and (4.20),

$$(2r-1)\begin{bmatrix} -1 & 1 \\ z & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{r-1} & 0 \\ 0 & \frac{1}{r} \end{bmatrix} M^{2r-1}(z)$$

$$= \binom{2r-1}{r}\begin{bmatrix} r & 0 \\ 0 & -1 \end{bmatrix} A(1)\left\{\prod_{\ell=1}^{r-2} B\left(\frac{2\ell}{r^2-\ell^2}\right) A(2\ell+1)\right\}$$

$$\times B\left(\frac{2r-2}{2r-1}\right)\begin{bmatrix} 1 & \frac{1}{2r-1} \\ (2r-1)z & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.21}$$

But in a similar manner to (4.16) and (4.17),

$$\begin{bmatrix} 1 & \dfrac{1}{2r-1} \\ (2r-1)z & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & p_{2r-1}(z) \\ 0 & (z-1)^{2r-1} \end{bmatrix}$$

$$= -A(2r-1) \begin{bmatrix} 1 & p_{2r}(z) \\ 0 & (z-1)^{2r} \end{bmatrix}. \tag{4.22}$$

So combining (4.18), (4.21), and (4.22) gives (4.9) for $k = 2r$, with M^k given by (4.10). ∎

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